A SIMPLE NEURON MODEL

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1. Basic Function of a Neuron

Neurons are cells in the body that transmit information to other neurons and surrounding cells via a synapse, which is a structure that facilitates the information exchange. The information transmission comes in two flavors, electrical and chemical. When a neuron receives an electrical charge input, commonly in the form of sodium ions, it amplifies the stimulus. If the amplification reaches a certain threshold, called an action potential, then the neuron fires resulting in a transfer of energy to the surrounding neurons. After the neurons fires, it becomes a resting neuron again by utilizing sodium and potassium pumps that allow excess potassium to exit the cell and decreasing the permeability of the membrane to sodium ions once again. The summary above is a generalization and does not contain many of the finer details occuring when a neuron actually fires. However our brief summary will be enough to understand the mathematical model that follows. More detail can be found in [1].

2. Derivation of the FitzHugh-Nagumo Model

What we would like in our model is a way to simulate the neuron's potential energy changing from it's resting potential to the action potential, and then back to the resting potential. This can be partly achieved by considering the following differential equation,

$$\frac{dv}{dt} = \dot{v} = -v(v-a)(v-1) \tag{2.1}$$

where v is the energy potential and $0 < a < 1$.

Notice this equation has roots at $v = 0, v = a$, and $v = 1$. We see that when $v = 0$ the neuron is at rest, which means the potassium, sodium, and chloride ions are positioned in favor of the neurons membrane potential. It can be observed that (2.1) has two stable fixed points at $v = 0$ and $v = 1$ and an unstable fixed point at $v = a$. Hence, if $v > a$ the solution will increase to $v = 1$ and if $v < a$ the solution will decrease to 0.

This provides a way for the neuron to increase its voltage, but (2.1) by itself does not suffice, since in reality the neuron spikes for only a moment before a blocking mechanism acts against the increase in potential and returns the potential to its resting state.

The blocking mechanism, w, is added to the system by imagining the subtraction of an increasing potential energy term from (2.1) as v increases. This term should depend on v, such that as v increases, w also increases. Such an object would allow v to reach the action potential, allowing the neuron to fire, and then facilitate the return of the neurons potential to its resting potential. Consider the following equation,

$$\frac{dw}{dt} = \dot{w} = \epsilon(v - \xi w) \tag{2.2}$$

where ϵ is a parameter that determines how fast the blocking mechanism acts, and for us ξ will typically be set to 1.

Now we modify equation (2.1) by subtracting w from it and we obtain the following coupled system, named the FitzHugh-Nagumo system,

$$\begin{cases} \dot{v} &= -v(v-a)(v-1) - w \\ \dot{w} &= \epsilon(v - \xi w) \end{cases} \tag{2.3}$$

Important behaviors of w can be obtained by considering some limiting cases, such as when $v = 0$ and $v = 1$. First, let $v = 0$, then we obtain $\dot{v} = -w = 0$. Furthermore if $v = 1$ we obtain $\dot{v} = -w = 0$ and $\dot{w} = \epsilon(1 - \xi w)$. It follows that $\frac{d^2v}{dt^2} = \dot{w} = 0$ which finally leads to $\dot{w} = \epsilon - \epsilon\xi w = 0 \implies w = \frac{1}{\xi}$ when $v = 1$. Thus we conclude that when the neuron is at rest the strength of the blocking mechanism is zero and when the potential increases to one the blocking mechanism has a maximum strength of $\frac{1}{\xi}$.

3. Equilibrium Analysis

To find the equilibria we set both equations in (2.3) equal to zero which yields,

$$\begin{aligned} \dot{v} &= -v(v-a)(v-1) - w = 0 \\ \implies w = &\ -v(v-a)(v-1) \end{aligned}$$

$$\begin{aligned} \dot{w} &= \epsilon(v - \xi w) = 0 \\ \implies w = &\ \frac{v}{\xi} \end{aligned}$$

From these two equations we see that $E_0 = (0, 0)$ is an equilibrium point by inspection. Furthermore, we can conclude it will be the only equilibrium point under the following condition. Assume $v \neq 0$ then,

$$\begin{aligned} \frac{v}{\xi} &= -v(v-a)(v-1) \\ \frac{1}{\xi} &= -(v-a)(v-a) \\ 0 &= v^2 - (1+a)v + \left(a + \frac{1}{\xi}\right) \end{aligned}$$

Solving for v yields $a^2 - 2a + 1 - \frac{4}{\xi}$ under the discriminant. Hence we can guarantee that E_0 is the only fixed point under the condition that $\xi < 4$, which it will be for the purposes of our analysis. This result can be seen geometrically by considering the nulliclines shown in Figure 3.1.

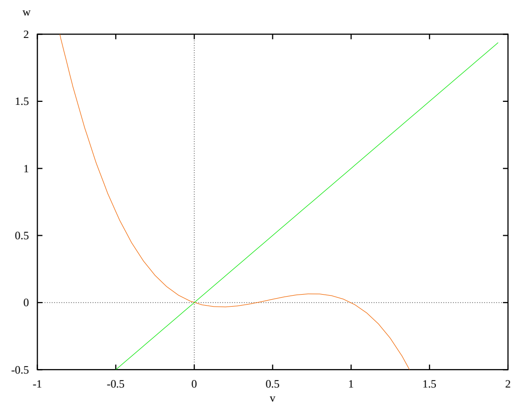


FIGURE 3.1. Nulliclines for (2.3) with $\epsilon = 0.1, \xi = 1, a = 0.4$.

Next we analyze the stability of E_0 by considering the Jacobian of (2.3). It yields the following,

$$\begin{aligned} J(E_i) &= \begin{pmatrix} \frac{\partial \dot{v}}{\partial v} & \frac{\partial \dot{v}}{\partial w} \\ \frac{\partial \dot{w}}{\partial v} & \frac{\partial \dot{w}}{\partial w} \end{pmatrix} \\ &= \begin{pmatrix} -3v^2 + 2(1+a)v - a & -1 \\ \epsilon & -\epsilon\xi \end{pmatrix} \end{aligned}$$

Since E_0 is the only equilbrium we obtain,

$$J(E_0) = \begin{pmatrix} -a & -1 \\ \epsilon & -\epsilon\xi \end{pmatrix}$$

To determine the stability of E_0 we calculate the trace and determinant of the matrix above which yields,

$$\begin{cases} Trace(J(E_0)) & = -a - \epsilon\xi < 0 \\ Det(J(E_0)) & = \epsilon(a\xi + 1) > 0 \end{cases}$$

The above results imply that E_0 is an asymptotically stable equilibrium point.

In Figure 3.2 we consider an orbit of (2.3). Notice how the system allows for a sharp increase in potential until the blocking mechanism reaches its maximum strength and similarly forces a sharp decrease in potential which forces the potential to overshoot zero before increasing back to zero. It is clear in this picture how the addition of the blocking mechanism allows for the neuron to reset instead of being permanently inactive or active, which was the case before we added the blocking mechanism. The Figure below 3.2 shows a plot of t vs. v which provides another way of viewing the potential as an object that has a sharp increase followed by a sharp decrease.

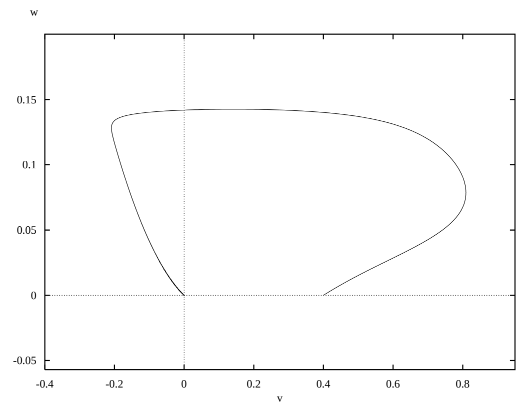
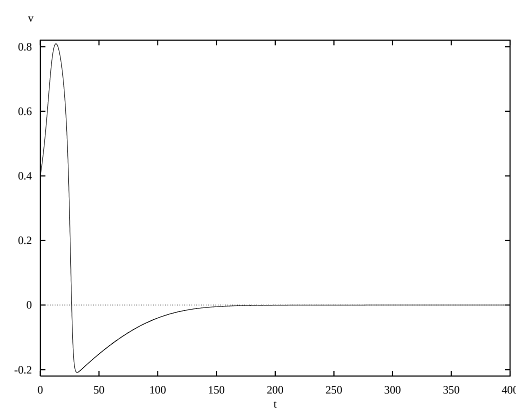


FIGURE 3.2. A typical solution with $a = 0.3$, $\xi = 1$, $\epsilon = 0.01$ and initial condition $(0.4, 0)$.



(A) The plot of t vs. v of the solution in Figure 3.2 above.

4. Adding a Constant Electrical Current

Consider the addition of a constant electrical current J to the rate of change of potential in (2.3) which yields the new system,

$$\begin{cases} \dot{v} &= -v(v-a)(v-1) - w + J \\ \dot{w} &= \epsilon(v - \xi w) \end{cases} \tag{4.1}$$

Notice that if J is too strong then it will overpower the blocking mechanism and change the stability of the system. In Figure 4.1 we demonstrate that the addition of J moves the equilibrium E_0 from the origin to the first quadrant.

We can observe this analytically by solving for $\dot{v} = 0$ and $\dot{w} = 0$ in (4.1) which yields the following equilibrium points,

$$\begin{cases} v &= \frac{w}{\xi} \\ w &= -v(v-a)(v-a) + J \end{cases}$$

Solving this system yields $v^2 - (1+a)v + (a+\xi) = \frac{-J}{v}$ which can intersect at most once in the first quadrant, hence shifting our equilibrium point.

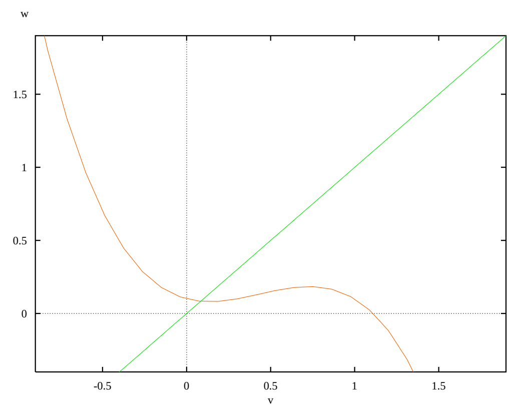


FIGURE 4.1. Nulliclines of (4.1) with $a = 0.3$, $\xi = 1$, $\epsilon = 0.01$,$J = 0.1$

We obtain some very interesting results when we vary the value of J. In Figure 5.1 (attached at the end) we vary J between 0 and 0.2 which yields many different types of equilibrium behavior. Initially we see the stable equilibria turn into a spiral sink and finally turn into a periodic orbit. This demonstrates that as J gets larger the equilibrium point becomes unstable. It is observed graphically that a J value in the neighborhood of 0.164 makes the equilibrium change from a stable endpoint to a periodic orbit.

The physical reason behind this occurence is because in that neighborhood the constant current begins to overpower the blocking mechanism which changes the systems ability to fire and drop back down to its resting potential.

5. CONCLUSION

It has been shown that the behavior of this relatively simple model is a very good approximation to a neuron. Furthermore, this model yields rich results because we can create conditions on a parameter to ensure only a single equilibrium point at the origin exists. This reduces the problem to the analysis of how the system behaves around just a single point, which is always a good thing. Finally we showed how the introduction of a constant current to the system will change the long term behavior of the solution by either overpowering the blocking mechanism, or by preserving the stability of the solution.

Some questions to think about are what happens if we defy the condition on ξ and allow for the addition of other equilibrium points. How would the solutions behave in such a system? Does J still alter the system in a similar way with the addition of these equilibrium points or does it behave differently? These are just some of the questions one could ask, and in [3] these questions and many more have been answered.

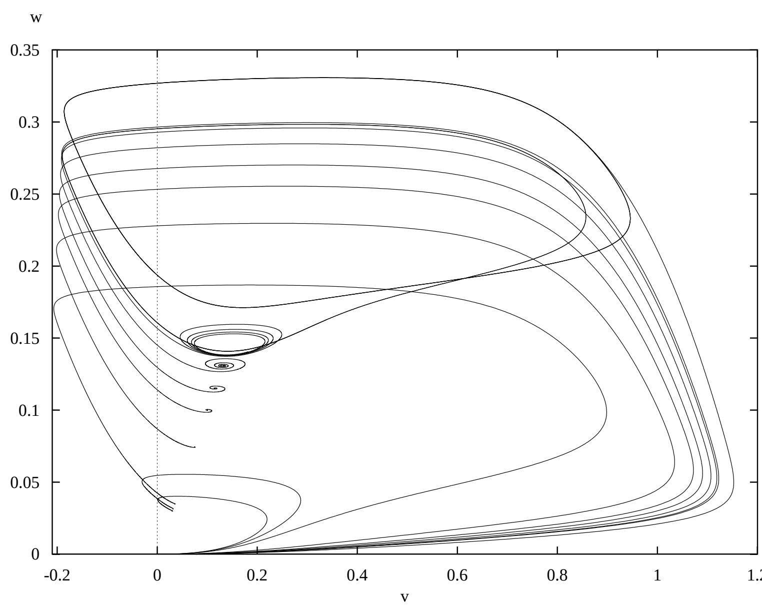


FIGURE 5.1. System (4.1) with $a = 0.3$, $\xi = 1$, $\epsilon = 0.01$,$0 < J < 0.2$ plotted with 10 J values in that range.

REFERENCES

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